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Controllability Issues in Nonlinear State-Dependent Riccati Equation Control

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In the last few years, algorithms using state-dependent Riccati equations (SDREs) have been proposed for solving nonlinear control problems. Under state feedback, pointwise solutions of an SDRE must be obtained along the system trajectory. To ensure the control is well defined, global controllability and observability of state-dependent system factorizations are commonly assumed. Here connections between controllability of the state-dependent factorizations and true system controllability are rigorously established. It is shown that a local equivalence always holds for the class of systems considered, and special cases that imply global equivalence are also given. Additionally, a notion of nonlinear stabilizability is introduced, which is a necessary condition for global closed-loop stability. The theory is illustrated by application to a five-state nonlinear model of a dual-spin spacecraft.

I. Introduction

SINCE the early 1960s, a number of researchers have proposed nonlinear control algorithms that involve application of linear design methods to linearlike factored representations of a nonlinear system.[1–7] For continuous time, state feedback, input-affine, autonomous nonlinear dynamic systems

$$\dot{\boldsymbol{x}} = \boldsymbol{a}(\boldsymbol{x}) + b(\boldsymbol{x})\boldsymbol{u}, \qquad \boldsymbol{a}(0) = 0$$
$$z = \begin{bmatrix} \boldsymbol{h}(\boldsymbol{x}) \\ \bar{R}(\boldsymbol{x})\boldsymbol{u} \end{bmatrix}, \qquad \boldsymbol{h}(0) = 0 \tag{1}$$

with state vector $\boldsymbol{x} \in \mathcal{R}^n$, control vector $\boldsymbol{u} \in \mathcal{R}^m$, penalized variable $z \in \mathcal{R}^s$, and nonsingular (for all $\boldsymbol{x}$) control penalty matrix function $\bar{R}(x)$, it is assumed that one can obtain linear-looking factored representations of the form

$$\boldsymbol{a}(\boldsymbol{x}) = A(\boldsymbol{x})\boldsymbol{x}, \qquad b(\boldsymbol{x}) = B(\boldsymbol{x}), \qquad \boldsymbol{h}(\boldsymbol{x}) = H(\boldsymbol{x})\boldsymbol{x} \tag{2}$$

This concept has alternatively been called apparent linearization,[3] extended linearization,[5] or most recently state-dependent coefficient factorization[6] of Eq. (1).

The earliest known proposal for using state feedback Riccati-based linear control methods on nonlinear systems appears in Ref. 1, motivated by finite time suboptimal regulation. In Ref. 1, the steady-state stabilizing solution to a state- and time-dependent Riccati differential equation is obtained analytically for low-order example systems to provide a basis for a suboptimal control algorithm. Global asymptotic stability for each example is proven, and suboptimal controller performance is shown to compare quite favorably with that of the corresponding optimal controllers. Variations on this approach are proposed in Refs. 2 and 3, which also involve solving an algebraic state-dependent Riccati equation (SDRE) for any location traversed in the state space. In Ref. 4, the same basic idea is revisited, and conditions relating the suboptimal solution to the optimal solution are derived. In both Refs. 3 and 4, the state and control weights $\bar{R}$ and H are assumed to be constant matrices, so that the regulation problem is indeed quadratic, whereas in Refs. 1 and 2, $\bar{R}$ and H are arbitrary time-varying matrices.

More recently, in Ref. 5 application of any linear control algorithm to Eq. (2) is suggested, but no theoretical justification for such an approach is given. In Ref. 6 both state and output feedback SDRE approaches to regulation and nonlinear H_∞ control problems are proposed, where weighting matrices are not restricted to be constants or functions of time but may instead be functions of $\boldsymbol{x}$. Local stability is proven for suboptimal state feedback versions of the approach, and an additional necessary condition that must be satisfied for optimality of the state feedback regulator is given. In Ref. 7 a Lyapunov function is proposed for establishing global stability of the suboptimal SDRE state feedback regulator, based on a restricted class of weighting matrix functions.

Although some progress has been made in theoretically justifying such linearized methods, much remains to be done. We develop some of the needed theory, by examining controllability issues in the context of the state feedback regulator problem. We derive conditions under which global factored and true nonlinear controllability hold and show how each type of controllability is separately important to successful application of the SDRE approach. These issues have not been addressed in the literature and have significant implications for both factorization selection and global stability of SDRE-based control algorithms.

The outline of the paper is as follows. In Sec. II we briefly review the SDRE nonlinear regulation control algorithm. We then establish connections between factored and true nonlinear controllability in Sec. III. In Sec. IV we formalize our results and illustrate the given theorems with simple examples. We then demonstrate the significance of our results by analyzing the impact of both types of controllability issues on SDRE nonlinear regulation of a more realistic dynamic model in Sec. V. Finally, we summarize and conclude in Sec. VI.

II. Control Algorithm

We assume that $\boldsymbol{a}$ and $\boldsymbol{h}$ in Eq. (1) are real-valued C^1 functions of $\boldsymbol{x}$ on $\mathcal{R}^n$. Under this assumption system (1) can be written[8] (nonuniquely) in the so-called state-dependent coefficient (SDC) form (2), where A and H are chosen to be (at least) continuous. The control algorithm may is as follows. Consider any initial condition $\boldsymbol{x}_0 \in \mathcal{R}^n$. The objective is near-optimal regulation, i.e., drive the state to zero while simultaneously keeping the cost function

$$J = \int_0^\infty z^T z \, \mathrm{d}t = \int_0^\infty \boldsymbol{x}^T Q(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{u}^T R(\boldsymbol{x})\boldsymbol{u} \, \mathrm{d}t \tag{3}$$

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close to its optimal value, where we have defined $R(\boldsymbol{x}) = \bar{R}^T(\boldsymbol{x})\bar{R}(\boldsymbol{x}) > 0$ and $Q(\boldsymbol{x}) = H^T(\boldsymbol{x})H(\boldsymbol{x}) \geq 0\ \forall\, \boldsymbol{x}$. This may be accomplished locally[6,7] setting

$$\boldsymbol{u}(\boldsymbol{x}) = -R^{-1}(\boldsymbol{x})B^T(\boldsymbol{x})P(\boldsymbol{x})\boldsymbol{x} \tag{4}$$

where $P(\boldsymbol{x})$ is the maximal, stabilizing solution to the algebraic continuous time SDRE

$$A^T(\boldsymbol{x})P(\boldsymbol{x}) + P(\boldsymbol{x})A(\boldsymbol{x}) - P(\boldsymbol{x})B(\boldsymbol{x})R^{-1}(\boldsymbol{x})B^T(\boldsymbol{x})P(\boldsymbol{x}) + Q(\boldsymbol{x}) = 0 \tag{5}$$

Now, to ensure the desired solution of Eq. (5) exists for all $\boldsymbol{x}$, one may assume the pairs $\{A(\boldsymbol{x}), B(\boldsymbol{x})\}$ and $\{H(\boldsymbol{x}), A(\boldsymbol{x})\}$ are controllable and observable, respectively, for all $\boldsymbol{x}$, where we employ the common definitions of controllability and observability from linear systems theory.[9] Of course, less restrictive assumptions such as stabilizability and detectability would also be sufficient. However, because stabilizability is defined based on controllability, we study the relationship between the factored controllability assumed to guarantee existence of solutions of Eq. (5) and the true nonlinear controllability of system (1).

III. Factored vs True Controllability

In previous studies of controllability, e.g., Ref. 10, researchers have established connections between factored and true system controllability in only the linear time-invariant (LTI) case, and it appears any other possible connections between the two have been left undeveloped. In this section we review the known results linking the two types of controllability and then proceed to develop additional cases in which the two are linked.

Comparison of factored vs true nonlinear controllability is facilitated by considering controllability in terms of the dimension of invariant locally reachable and unreachable spaces. For controllable systems, the dimension of the locally reachable space must equal the dimension of the state space. Thus, for LTI systems, controllability is established[11] by verifying that the rank of the controllability matrix M_{cl} equals n, where $M_{cl} = [B\ \ AB\ \ A^2B\ \ \cdots\ \ A^{n-1}B]$. For the SDC factored system (2), this test generalizes to a rank test on the factored controllability matrix function

$$\operatorname{rank}[M_{cf}(\boldsymbol{x})] = \operatorname{rank}[B(\boldsymbol{x})\quad A(\boldsymbol{x})B(\boldsymbol{x})\quad A^2(\boldsymbol{x})B(\boldsymbol{x})\quad \cdots\quad A^{n-1}(\boldsymbol{x})B(\boldsymbol{x})] = n\ \forall\, \boldsymbol{x} \tag{6}$$

whereas for the original system, local controllability is characterized[10] at each $\boldsymbol{x}$ in terms of the dimension of the span of the smallest nonsingular and involutive distribution $\Delta_c(\boldsymbol{x})$ containing the columns b_i of $B(\boldsymbol{x})$, $1 \leq i \leq m$, and invariant under a and the b_i. This distribution assigns to each $\boldsymbol{x} \in \mathcal{R}^n$ a vector space, an open subset of which is reachable from the given point by using piecewise constant inputs. Thus, a sufficient condition[10] for system (1) to be locally controllable at the point $\boldsymbol{x}$ is

$$\operatorname{rank}[\Delta_c(\boldsymbol{x})] = n \tag{7}$$

and the system is said to be weakly controllable (on S) if Eq. (7) holds for all $\boldsymbol{x}$ $(\in S)$. The following recursive algorithm for generating Δ_c is given in Ref. 10.

1) Let $\Delta_0 = \operatorname{span}(B) = \operatorname{span}(b_i)$.

2) Let $\Delta_1 = \Delta_0 + [a, b_i] + [b_j, b_i],\ 1 \leq i \leq m,\ 1 \leq j \leq m$, where $[a, g]$ is the Lie bracket of a and g, i.e., $[a, g] = (\partial g/\partial x)a - (\partial a/\partial x)g$, and the $+$ indicates the sum of the spans.

3) Let $\Delta_k = \Delta_{k-1} + [a, d_j] + [b_i, d_j],\ 1 \leq i \leq m,\ 1 \leq j \leq n$, where $\{d_j\}$ is a basis for Δ_{k-1}.

4) Terminate when $\Delta_{k+1} = \Delta_k$.

For LTI systems, the listed procedure gives $\Delta_c = M_{cl}$ (Ref. 10). Also, Δ_{n-1} always equals Δ_c on an open and dense subset of $\mathcal{R}^n$, and if each Δ_k is nonsingular, then $\Delta_{n-1} = \Delta_c$ for all $\boldsymbol{x}$.

If a system fails the appropriate controllability rank test, then the uncontrollable subspace may be determined[10] by finding the annihilator of the appropriate matrix, i.e., the left nullspace of M_{cl}, M_{cf}, or Δ_c. Also, the controllable and uncontrollable subspaces as determined from M_{cl} and Δ_c are guaranteed to be invariant subspaces, whereas the respective subspaces determined from M_{cf} hold only for the single $\boldsymbol{x}$ value being considered. Thus, invariance of these pointwise sets is not ensured.

The described concepts allow controllable/uncontrollable system decompositions.[10,11] Thus, for an LTI system with controllable subspace of dimension d, we may write

$$\dot{\boldsymbol{x}}_1 = A_{11}\boldsymbol{x}_1 + A_{12}\boldsymbol{x}_2 + B_1\boldsymbol{u}, \qquad \dot{\boldsymbol{x}}_2 = A_{22}\boldsymbol{x}_2 \tag{8}$$

where $\boldsymbol{x}_1 \in \mathcal{R}^d$ is in the controllable subspace $\mathcal{C}_l$, $\boldsymbol{x}_2 \in \mathcal{R}^{n-d}$ is in the uncontrollable subspace $\mathcal{U}_l$, and $\{A_{11}, B_1\}$ is a controllable pair. For the nonlinear system (1) with controllable subspace of dimension d, we similarly may write

$$\dot{\boldsymbol{x}}_1 = \boldsymbol{a}_1(\boldsymbol{x}_1, \boldsymbol{x}_2) + b_1(\boldsymbol{x}_1, \boldsymbol{x}_2)\boldsymbol{u} \tag{9}$$

$$\dot{\boldsymbol{x}}_2 = \boldsymbol{a}_2(\boldsymbol{x}_2) \tag{10}$$

where $\boldsymbol{x}_1 \in \mathcal{R}^d$ is in the controllable subspace $\mathcal{C}_{nl}$, $\boldsymbol{x}_2 \in \mathcal{R}^{n-d}$ is in the uncontrollable subspace $\mathcal{U}_{nl}$, and Eq. (9) satisfies rank $[\Delta_c] = d$. Finally, Eqs. (8–10) allow us to discuss the notion of stabilizability. Conceptually, a system is stabilizable if its uncontrollable part is stable. For LTI systems, stabilizability is equivalent to A_{22} in Eq. (8) being a Hurwitz matrix, whereas nonlinear stabilizability requires that the unaffectable subsystem (10) be stable. Clearly, if a system may be written as Eqs. (9) and (10), for it to be drivable to the origin, this latter nonlinear stabilizability condition must hold, regardless of the factored controllability properties of the system.

IV. Theorems and Examples

With some minor extensions, the discussion of Sec. III allows us to give some basic theorems, the first of which is the lack of a general equivalency between factored and true nonlinear controllability. For purposes of space, proofs are sketched, and the reader is referred to Ref. 12 for details.

Theorem 1: Consider system (1) with $\boldsymbol{a}(\boldsymbol{x})$ and $\boldsymbol{h}(\boldsymbol{x})$ assumed to be C^1 functions, so that Eq. (1) may be written as in Eq. (2). Assume the pair $\{A(\boldsymbol{x}), B(\boldsymbol{x})\}$ is controllable for all $\boldsymbol{x}$, so that Eq. (6) holds. Then system (1) is not necessarily weakly controllable (on $\mathcal{R}^n$).

Proof: The proof is by counterexample. Consider the system

$$\dot{x}_1 = x_1x_2 + x_2, \qquad x_2 = u \tag{11}$$

With $A(\boldsymbol{x})$ chosen to be

$$A(\boldsymbol{x}) = \begin{bmatrix} x_2 & 1 \\ 0 & 0 \end{bmatrix} \tag{12}$$

It is easily verified that M_{cf} is globally full rank, but Δ_c as defined in Sec. III is such that at $x_1 = -1$, rank $[\Delta_c] = 1$, so that Eq. (7) fails to hold. □

The proof of theorem 1 shows that it is possible for a state-dependent factorization to hide the existence of an uncontrollable, invariant set [the set of all $x \in \mathcal{R}^2$ such that $\boldsymbol{x}_1 = -1$ for Eq. (11)]. It is interesting to note that this fact is not mentioned in Ref. 6, nor, it appears, has it historically been well known when SDRE type methods were previously suggested. In fact, in Ref. 3 the factorization (12) is recommended as a better factorization for Eq. (11) than the choice

$$A(\boldsymbol{x}) = \begin{bmatrix} 0 & x_1 + 1 \\ 0 & 0 \end{bmatrix} \tag{13}$$

because Eq. (12) allows solution for the control at all $\boldsymbol{x}$, whereas Eq. (13) does not. Thus, the lack of true controllability of this system was not recognized by the authors of Ref. 3. By defining J as the Jacobian of a, i.e., $J(\boldsymbol{x}) = \partial \boldsymbol{a}/\partial \boldsymbol{x}$, the following theorem may be proven.

Theorem 2: Consider system (1) written as Eq. (2) with $n = 2$, and let B be a constant matrix. Also, assume that $A(\boldsymbol{x})$ is chosen such that $J(\boldsymbol{x})B = kA(\boldsymbol{x})B\ \forall\, \boldsymbol{x}$, where $0 \neq k \in \mathcal{R}$. Then, if the factorization (2) is controllable for all $\boldsymbol{x}$, system (1) is weakly controllable on an open and dense subset of $\mathcal{R}^2$. Conversely, if Eq. (1) is weakly controllable on $\mathcal{R}^2$, then Eq. (2) is controllable for all $\boldsymbol{x}$.

Proof: The proof follows by checking the rank of M_{cf} and Δ_c under the assumptions. □

We now give some corollaries to theorem 2, which are easily proven and useful when considering factorization choices.

Corollary 1: The assumption $T_A(\boldsymbol{x})\boldsymbol{x}B = (J - A)B = (k-1)AB$, $k \neq 0$, is equivalent to the assumption $J(\boldsymbol{x})B = kA(\boldsymbol{x})B$, $k \neq 0$, where $T_A(\boldsymbol{x}) = (\partial A/\partial \boldsymbol{x})$.

Corollary 2: $T_A(\boldsymbol{x})\boldsymbol{x}B \neq -AB$ for all $\boldsymbol{x}$ is necessary for the assumptions of theorem 2 to hold.

Corollary 3: If B is not rank n, then $AB = 0$ iff $JB = 0$ is a necessary condition for the contrapositive of theorem 2 to hold.

We note that the conditions of the theorem 2 are sufficient, but not necessary. It is possible for both M_{cf} and Δ_c to be full rank without AB being in the range of JB alone, as long as both AB and JB provide the remainder of a spanning set of $\mathcal{R}^2$ not provided by B. In theorem 2 we restrict attention to second-order systems because for higher-order systems, more iterations on Δ_k and, thus, more Lie bracket calculations will generally be required. The divergence between succeeding entries in M_{cf} and Δ_c increases as n increases, making useful comparisons between the two more difficult. However, that $T_A(\boldsymbol{x})\boldsymbol{x} + A(\boldsymbol{x}) = J(\boldsymbol{x})$ allows us to draw the following conclusion regarding factored and local nonlinear controllability for any n.

Theorem 3: Consider system (1) written as Eq. (2), and assume Eq. (6) holds. Then system (1) is weakly controllable on some local neighborhood of the origin.

Proof: The conclusion follows because $J(0) = A(0)$ implies $M_{cf}(0) = M_{cl}$, where M_{cl} is in terms of the system linearization, and the assumptions guarantee $M_{cf}(0)$ is full rank. □

The set where system (1) is weakly controllable is, thus, the set on which its linearization dominates, which may be arbitrarily small, as illustrated later.

The preceding three theorems dealt with an equivalence relationship between M_{cf} and Δ_c involving assumptions on $A(\boldsymbol{x})$ or on the dimension of the state. The following (trivial) theorem gives one case in which such assumptions are unnecessary.

Theorem 4: Consider system (1) written as Eq. (2) and let $m \geq n$. Assume $B(\boldsymbol{x})$ has rank n for all $\boldsymbol{x}$. Then Eq. (2) is controllable for all $\boldsymbol{x}$, and system (1) is weakly controllable on $\mathcal{R}^n$.

Proof: The proof follows trivially from constructing M_{cf} and Δ_c, noting $B(\boldsymbol{x})$ satisfies by itself the rank requirement. □

The importance of theorem 4 is that pointwise controllability issues are not a driving force in factorization choices for a when B is globally rank n, as they definitely are when this is not the case. We now illustrate the theorems with some examples.

Example 1: Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_1 x_2 + u \tag{14}$$

We have $\boldsymbol{a} = [x_2 \;\; x_1x_2]^T$, $\boldsymbol{b} = [0 \;\; 1]^T$, and because $m < n$, theorem 4 may not be used. However, we have

$$J = \begin{bmatrix} 0 & 1 \\ x_2 & x_1 \end{bmatrix}, \qquad \Delta_c = \Delta_1 = \begin{bmatrix} 0 & 1 \\ 1 & x_1 \end{bmatrix} \tag{15}$$

so that Δ_c has rank 2 for all $\boldsymbol{x}$. Now, choose

$$A(\boldsymbol{x}) = \begin{bmatrix} 0 & 1 \\ 0 & x_1 \end{bmatrix} \Rightarrow A(\boldsymbol{x})B = \begin{bmatrix} 1 \\ x_1 \end{bmatrix} = JB \; \forall \boldsymbol{x} \tag{16}$$

so that the conditions of theorem 2 are satisfied. Checking the factored controllability matrix we find

$$M_{cf}(\boldsymbol{x}) = \begin{bmatrix} 0 & 1 \\ 1 & x_1 \end{bmatrix} = \Delta_c(\boldsymbol{x}) \; \forall \boldsymbol{x} \tag{17}$$

and we see that the system is weakly controllable on $\mathcal{R}^2$, whereas the factored system is controllable for all $\boldsymbol{x} \in \mathcal{R}^2$ as well. On the other hand, if we choose

$$A(\boldsymbol{x}) = \begin{bmatrix} x_2 & 1 - x_1 \\ 0 & x_1 \end{bmatrix} \tag{18}$$

then the factored controllability matrix function

$$M_{cf}(\boldsymbol{x}) = \begin{bmatrix} 0 & 1 - x_1 \\ 1 & x_1 \end{bmatrix} \tag{19}$$

clearly loses rank at $x_1 = 1$. Given the preceding analysis, the first choice of $A(\boldsymbol{x})$ for this example is preferable because it guarantees global existence of the control. Thus, even when the original system is weakly controllable, care must be taken when choosing the factorization, as both poor and good choices from an implementation standpoint may exist.

In the next example we illustrate the nonnecessity of the $AB = kJB$ condition as discussed earlier.

Example 2: Consider the system

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = x_1^2 + x_2^2 + u \tag{20}$$

Theorem 4 may not be invoked, but simple computations give

$$J = \begin{bmatrix} 0 & 1 \\ 2x_1 & 2x_2 \end{bmatrix}, \qquad \Delta_c = \Delta_1 = \begin{bmatrix} 0 & 1 \\ 1 & 2x_2 \end{bmatrix} \tag{21}$$

so that Δ_c has rank 2 for all $\boldsymbol{x}$. Now, choose

$$A(\boldsymbol{x}) = \begin{bmatrix} 0 & 1 \\ x_1 & x_2 \end{bmatrix} \Rightarrow A(\boldsymbol{x})B = \begin{bmatrix} 1 \\ x_2 \end{bmatrix} \tag{22}$$

so that AB does not equal a multiple of JB for any $\boldsymbol{x}$. However, checking the factored controllability matrix we find

$$M_{cf}(\boldsymbol{x}) = \begin{bmatrix} 0 & 1 \\ 1 & x_2 \end{bmatrix} \tag{23}$$

which is full rank for all $\boldsymbol{x}$, so that global weak and factored controllability both hold. Thus, even though theorem 2 does not apply, $[1 \;\; 2x_2]^T$ and $[1 \;\; x_2]^T$ both provide vector functions, which, together with $\boldsymbol{b} = [0 \;\; 1]^T$, span $\mathcal{R}^2$ for all values of $\boldsymbol{x}$.

Finally, we return to the counterexample in the proof of theorem 1, and see how corollary 2 comes into play.

Example 3: Consider the system

$$\dot{x}_1 = x_1 x_2 + x_2, \qquad \dot{x}_2 = u \tag{24}$$

We have rank $(B) = 1 \; \forall \boldsymbol{x}$, and again theorem 4 may not be invoked. We find

$$J = \begin{bmatrix} x_2 & x_1 + 1 \\ 0 & 0 \end{bmatrix}, \qquad \Delta_c = \begin{bmatrix} 0 & x_1 + 1 \\ 1 & 0 \end{bmatrix} \tag{25}$$

so that Δ_c has rank 1 at $x_1 = -1$ as we saw before. Now, also as before, choose

$$A(\boldsymbol{x}) = \begin{bmatrix} x_2 & 1 \\ 0 & 0 \end{bmatrix} \Rightarrow A(\boldsymbol{x})B = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \tag{26}$$

which gives the globally full rank factored controllability matrix function

$$M_{cf}(\boldsymbol{x}) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \tag{27}$$

However, we see that kJB does not equal AB for $\boldsymbol{x} = -1$ for any $k \neq 0$. Also, $T_A\boldsymbol{x}B = (J - A)B = -AB$ when $x_1 = -1$, and we have $AB \neq 0$ when $JB = 0$, both of which we noted were necessary conditions for equivalency of the controllability tests if B were not rank n. Thus, the factored system is controllable for all $\boldsymbol{x} \in \mathcal{R}^2$, whereas the original system is not weakly controllable on $x_1 = -1$. Finally, this example illustrates the potential weakness of theorem 3. Clearly, the controllability equivalence based on linearization does not hold in this example beyond a ball of radius one centered at the origin. By replacing the first element of a with $x_1x_2 + kx_2$, with $k \in \mathcal{R}$, and decreasing the absolute value of k, we can construct an example for which the conclusion of theorem 3 holds on an arbitrarily small neighborhood of the origin. Thus, the study of global as opposed to local controllability equivalence is indeed well motivated.

To this point we have shown that both factored and true nonlinear controllability are separately important to the success of the SDRE approach. Factored controllability is important from a computational standpoint, whereas true controllability has ramifications for stability. Although we have shown these concepts are basically different, we have established some conditions, particularly for low-order systems, when they simultaneously hold, and additionally have established implications from one form of controllability to the other. We now extend our study of these relationships by considering controllability issues in a nontrivial design example, showing how the desired properties may be established.

V. Design Problem

The chosen problem involves angular momentum control of an axial dual-spin spacecraft and exhibits highly nonlinear dynamics and limited controllability. Thus, it well serves to illustrate the theory. In the sequel we first describe the design problem and then give the equations of motion and design objectives. Following this we present a brief controllability analysis of the open-loop system, from both the factored and true nonlinear perspectives. Simulation results demonstrate successful application of the method.

A. Problem Description

A dual-spin satellite consists of two bodies capable of relative rotation, with one body spinning relatively fast (the rotor) to provide stabilization and one body (the platform) spinning relatively slowly to perform mission requirements, i.e., to remain Earth pointing.[13] Typical deployment scenarios result in both bodies initially spinning at nearly the same rate about a single axis (the so-called all-spun condition), so that some type of spin-up maneuver is required to despin the platform. Spin-up maneuvers have been investigated by numerous researchers as described in Refs. 14 and 15. Typical maneuvers employ a small, constant internal torque applied to the rotor. In this paper we apply the SDRE nonlinear regulation technique to the so-called transverse spin-up maneuver. This maneuver, also known as the dual-spin turn, has initial conditions such that the spacecraft spins about a principal axis nearly perpendicular to the rotor spin axis and a desired final condition such that all the spacecraft's angular momentum is contained in the rotor.

The model used in Refs. 13 and 14 is an axial gyrostat, comprising a rigid asymmetric platform and a rigid axisymmetric rotor constrained to relative rotation about its symmetry axis. The rotational dynamics equations for this model may be written in terms of four states: the three components of the system angular momentum and the axial component of the rotor angular momentum. Simple analysis of the equations of motion reveals that the numerical SDRE regulation approach may not be used to stabilize this system. This is because the (unique) system linearization about the desired equilibrium point is not stabilizable in the linear sense and, therefore, stabilizing solutions to the SDRE do not exist near the origin.

Remark: These results point out a limitation of numerical SDRE methods: For systems with linearly uncontrollable imaginary open-loop eigenvalues in their linearizations, it may be possible to nonlinearly stabilize the corresponding modes (as controllability analysis of the single-rotor gyrostat would indicate). The numerical SDRE regulator, however, reverts to standard linear quadratic regulation near the origin, and so will not, in general, be stabilizing for such systems.

In light of the preceding discussion we investigate a two-rotor gyrostat model where the second (uncontrolled) rotor is subject to an internal viscous damping torque. As shown in the sequel, the addition of this second rotor results in a stabilizable system linearization at the origin.

B. Equations of Motion

A general version of the two-rotor gyrostat model used here is developed in detail in Ref. 15. Referring to Fig. 1, we define the vector $\boldsymbol{x} \in \mathcal{R}^3$ as the gyrostat angular momentum vector expressed in the body-fixed principal reference frame (e_1, e_2, e_3), noting that e_1 is the axis about which the platform and controlled rotor may have relative rotation (the spin axis). We neglect external torques so that $\|\boldsymbol{x}\|$ is constant, assumed nonzero, and scaled so that $\|\boldsymbol{x}\| = 1$. We define the scalar μ to be the controlled rotor axial angular momentum (about the e_1 axis) and the control input u to be the associated applied rotor torque. For the specific choice of the off-axis rotor spin axis being in the e_2 direction, we let the associated rotor axial angular momentum be μ_2. We also define dimensionless inertia parameters

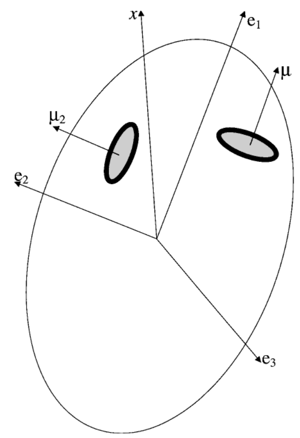

Fig. 1 Gyrostat model of two-rotor spacecraft.

i_j via $i_j = 1 - I_p/I_j$, $j = 1, 2, 3$, where $I_p = I_1 - I_s$, where I_s is the axial moment of inertia of the controlled rotor and I_j are the principal moments of inertia of the gyrostat. Using these definitions, the equations of motion may be written

$$\dot{x}_1 = (i_2 - i_3)x_2x_3 + \alpha_2\mu_2x_3, \qquad \dot{x}_2 = (i_3x_1 - \mu)x_3$$
$$\dot{x}_3 = -(i_2x_1 - \mu)x_2 - \alpha_2\mu_2x_1 \tag{28}$$
$$\dot{\mu}_2 = e_2x_2 - d_2\mu_2, \qquad \dot{\mu} = u$$

where

$$\alpha_2 = 1/I_2, \qquad e_2 = \alpha_2 I_{s_2}, \qquad d_2 = 1 + e_2 \tag{29}$$

and I_{s_2} is the off-axis rotor principal moment of inertia with respect to the e_2 axis.

Note that i_1 does not appear in the system equations but does impact the system via initial conditions. In the all-spun condition, this is evident because the initial condition for μ is $\mu = i_1\boldsymbol{x}_1$. The relationship between the other two inertia parameters determines whether the spacecraft is oblate or prolate.[14] Finally, the cone or nutation angle η (the angle between the e_1 axis and x) is defined via $\eta = \arccos x_1$.

The design goal of performing transverse spin-up maneuvers may be related to the state variables in Eq. (28) in the following way. In the transverse spin-up maneuver, we start with $x_1 \approx 0$ and $\mu = i_1x_1$, and we desire to drive the state to $(x_1, \mu) = (1, 1)$, noting that if we achieve this objective, then $x_2 = x_3 = 0$. The control problem, as posed, is actually a constrained nonzero setpoint problem. We, therefore, transform the problem into a form more suitable for SDRE regulation. In the next section we show how factorization controllability considerations motivate removal of the x_1 state equation, which we do to obtain an unconstrained problem. We address the nonzero setpoint problem of driving μ to 1 by the change of coordinates $\nu = \mu - 1$. This part of the control problem then reduces to regulation of ν, which fits readily into the SDRE framework. Finally, by recognizing $\|\boldsymbol{x}\| = 1$ always holds, we attempt only the regulation of (x_2, x_3) to $(0, 0)$, using the constraint to make $|x_1| = 1$. That x_1 remains free to take on either of the values $+1$ or -1 is a disadvantage of this approach. Simulation results verify success of the method, however, in that if we start with x_1 small but positive, we drive x_1 to $+1$ as desired.

C. Open-Loop/Controllability Analysis

Before developing the control design, we study the open-loop model (28) and its relevant controllability properties. Performing on Eq. (28) the iterative procedure for determining the control Lie algebra described in Sec. III, one can identify the uncontrollable coordinate $\phi(\boldsymbol{x})$ in the state space

$$\phi = c\left(x_1^2 + x_2^2 + x_3^2\right) \tag{30}$$

where c is an arbitrary constant. This uncontrollable coordinate corresponds to conservation of angular momentum, so that $\boldsymbol{x}$ trajectories remain on the unit momentum sphere centered at $(x_1, x_2, x_3) = (0, 0, 0)$. This uncontrollable coordinate, by itself, does not prevent us from reaching the desired equilibrium state because it lies on the sphere and, thus, does not imply that the system is not nonlinearly stabilizable. However, when $x_2 = x_3 = 0$, the uncontrollable space has dimension four and, in fact, consists of the (x_1, x_2, x_3, μ_2) space. This means that the equilibrium surfaces $(\pm 1, 0, 0, 0, \mu)$ have unchanging $\boldsymbol{x}$ components, regardless of how we select the control. The implication is that, if we seek to drive the system to $x_1 = 1$, we should avoid trajectories passing near the equilibrium at $x_1 = -1$, and vice versa. Complete analysis of the control Lie algebra[12] reveals no other invariant, uncontrollable manifolds with which we need to be concerned.

Because we have identified one globally uncontrollable coordinate, we seek to eliminate it and proceed with control design on the reduced system. To this end we define the coordinate transformation $q_1 = x_3$, $q_2 = x_2$, and $q_3 = \phi$, with ϕ defined as in Eq. (30) with $c = 0.5$, and eliminate x_1 as an independent variable by writing

$$x_1(q) = \pm\sqrt{1 - q_1^2 - q_2^2} \tag{31}$$

Because the Jacobian of this mapping is

$$J = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ x_1 & x_2 & x_3 \end{bmatrix} \tag{32}$$

we see by the inverse function theorem that the mapping is not one-to-one in a neighborhood of $x_1 = 0$. Thus, we need to know which hemisphere of the momentum sphere we are in to complete the mapping, so as to choose the appropriate sign in Eq. (31). With this coordinate change, the equations of motion become

$$\dot{q}_1 = (\mu - i_2 x_1) q_2 - \alpha_2 x_1 \mu_2, \qquad \dot{q}_2 = (i_3 x_1 - \mu) q_1$$
$$\dot{\mu}_2 = e_2 q_2 - d_2 \mu_2, \qquad \dot{\mu} = u \tag{33}$$

where $\boldsymbol{x}_1$ is defined as in Eq. (31), and we have eliminated the trivial state equation $\dot{q}_3 = 0$. The nonlinear controllability procedure for this system indicates lack of controllability of the $\boldsymbol{x}$ states only at the desired equilibrium $(q_1, q_2, \mu_2) = (0, 0, 0)$, provided $\boldsymbol{x}$ remains in the positive momentum sphere $(x_1 > 0)$. Recall, however, that we actually need to shift the state to make the desired closed-loop equilibrium point the origin, as discussed in Sec. V.B. To do so we define

$$\nu = \mu - 1 \tag{34}$$

leading to the system equations

$$\dot{q}_1 = (\nu - i_2 x_1) q_2 + q_2 - \alpha_2 x_1 \mu_2, \qquad \dot{q}_2 = (i_3 x_1 - \nu) q_1 - q_1$$
$$\dot{\mu}_2 = e_2 q_2 - d_2 \mu_2, \qquad \dot{\nu} = u \tag{35}$$

For Eq. (35) we find the Jacobian at zero to be

$$J(0) = \begin{bmatrix} 0 & 1 - i_2 & -\alpha_2 & 0 \\ i_3 - 1 & 0 & 0 & 0 \\ 0 & e_2 & -d_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{36}$$

so that stabilizability of the linearization requires that the eigenvalues of the upper left 3×3 subblock of Eq. (36) have negative real parts. We have nondimensionalized the spacecraft moments of inertia by assuming $I_p = 1$. This gives the simple equalities $1 - i_2 = \alpha_2$ and $i_3 - 1 = -1/I_3 \equiv -\alpha_3$, so that the characteristic equation of this subblock is

$$\lambda^3 + d_2\lambda^2 + \alpha_2\alpha_3\lambda + \alpha_2\alpha_3 = 0 \tag{37}$$

Using Routh's criterion[16] it is easy to show that the linearization of Eq. (35) is stabilizable. Linearized detectability is also guaranteed, provided ν has an independent, globally positive definite penalty, i.e., H is of the form

$$H = \begin{bmatrix} \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ \cdot & \cdot & \cdot & 0 \\ 0 & 0 & 0 & \rho(\boldsymbol{x}) \end{bmatrix} \tag{38}$$

where $\rho(\boldsymbol{x})$ is a positive definite function.

The preceding discussion shows that adding the off-axis damping rotor guarantees stabilizability and detectability in a neighborhood of the origin, for any parameterization A and for a suitable parameterization H. We are interested, however, in performing transverse spin-up maneuvers that start far away from the origin. We, thus, propose a factorization A that guarantees pointwise stabilizability and detectability everywhere in the positive momentum sphere $(x_1 > 0)$. Because ν is completely controllable, heuristically we want the factorization to show strong pointwise linear controllability of q_1 and q_2 through ν. We, therefore, choose

$$A(\boldsymbol{x}) = \begin{bmatrix} 0 & 1 - i_2 x_1 & -\alpha_2 x_1 & q_2 \\ i_3 x_1 - 1 & 0 & 0 & -q_1 \\ 0 & e_2 & -d_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{39}$$

This factorization has the factored controllability matrix function

$$M_{cf}(\boldsymbol{x}) = \begin{bmatrix} 0 & q_2 & -\xi_2 q_1 & \xi_1\xi_2 q_2 + \alpha_2 e_2 x_1 q_1 \\ 0 & -q_1 & \xi_1 q_2 & -\xi_1\xi_2 q_1 \\ 0 & 0 & -e_2 q_1 & e_2(\xi_1 q_2 + d_2 q_1) \\ 1 & 0 & 0 & 0 \end{bmatrix} \tag{40}$$

where

$$\xi_1 = i_3 x_1 - 1, \qquad \xi_2 = 1 - i_2 x_1 \tag{41}$$

The matrix function (40) has determinant $\det[M_{cf}] = -e_2[\xi_1^2 q_2^3 + d_2\xi_1 q_2^2 q_1 - \xi_1\xi_2 q_2 q_1^2 + (e_2\alpha_2 x_1 - d_2\xi_2) q_1^3]$ and obviously loses rank for nontrivial values of q_1 and q_2, so that Eq. (39) does not yield a globally controllable parameterization. However, Eq. (39) does yield guaranteed stabilizability for $x_1 > 0$, as can be shown using Routh's criterion and some simple analysis. For stabilizability of $\{A, B\}$, recall that we must have $y^T B \neq 0$ for all λ and y such that $y^T A = y^T \lambda$ and $\mathrm{Re}\lambda \geq 0$ (Ref. 9). The characteristic equation of Eq. (39) is

$$\lambda\left(\lambda^3 + d_2\lambda^2 - \xi_1\xi_2\lambda + \xi_1\xi_3\right) = 0 \tag{42}$$

where ξ_1 and ξ_2 are as in Eq. (41) and $\xi_3 = \alpha_2 e_2 x_1 - \xi_2 d_2$. We, thus, have a single-zero eigenvalue, plus three more eigenvalues determined by the roots of the term in parentheses in Eq. (42). For the zero eigenvalue, the corresponding left eigenvector is $y^T = [0\ 0\ 0\ 1]$. Because $y^T B = 1 \neq 0$, this zero eigenvalue is stabilizable. We now show that the remaining three eigenvalues have negative real parts under some slight additional assumptions. Recall that a necessary condition for only left half-plane roots of a polynomial is that all of the coefficients have the same sign.[16] By definition, $d_2 > 0$, and because $x_1 \leq 1$ and i_2 and $i_3 < 1$, we also have from Eq. (41) that $\xi_1 < 0$ and $\xi_2 > 0$, so that $-\xi_1\xi_2 > 0$. The final necessary condition, thus, becomes $\xi_3 < 0$. To complete the analysis, we assume without loss of generality a prolate spacecraft,[17] for which $i_2 > i_3 > 0$. From Eq. (41) we see that, for $x_1 > 0$, we have

$$\xi_3 < \alpha_2 e_2 + (i_2 - 1)d_2 = \alpha_2 e_2 - \alpha_2 d_2 = \alpha_2(e_2 - d_2) = -\alpha_2 < 0 \tag{43}$$

Thus, $\xi_3 < 0$, and we satisfy the necessary condition for left half-plane eigenvalues regardless of the values of α_2 and I_{s_2}. However, we still need to verify a sufficient condition for stability of these eigenvalues. Routhian analysis leads to the additional condition

$$-\xi_1(d_2\xi_2 + \xi_3) > 0 \tag{44}$$

Because $\xi_1 < 0$, using the definition of ξ_3 we find that Eq. (44) becomes

$$d_2\xi_2 + \alpha_2 x_1 e_2 - \xi_2 d_2 = \alpha_2 x_1 e_2 > 0 \tag{45}$$

which holds as long as $x_1 > 0$, as assumed throughout. Thus, Eq. (39) gives a pointwise stabilizable parameterization, as long as trajectories remain in the positive (x_1) momentum sphere.

Remark: The existence of this stabilizable parameterization is interesting, inasmuch as a manifold of open-loop equilibrium points with unstabilizable local linearizations is known to exist[18] in the positive hemisphere for the prolate case. If a trajectory were to pass through this manifold, then a control algorithm based on Riccati equations and local linearizations such as linear matrix inequality-based techniques would fail to yield computable controllers at such points, whereas the SDRE control is still well defined.

Note that Eq. (39) corresponds to the choices of $c_1 = c_2 = 0$ in the general factorization

$$A(\boldsymbol{x}) = \begin{bmatrix} 0 & 1 - i_2 x_1 + c_1 \nu & -\alpha_2 x_1 & (1 - c_1) q_2 \\ i_3 x_1 - 1 - c_2 \nu & 0 & 0 & -(1 - c_2) q_1 \\ 0 & e_2 & -d_2 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{46}$$

and observe that, if we pick $c_1 = c_2 = 1$, then the q dynamics appear to be pointwise unaffectable from ν for all $\boldsymbol{x}$. In this case the uncontrollable space is all of the (q_1, q_2, μ_2) space, as opposed to the much smaller uncontrollable space of Eq. (39). Stabilizability of the factorization (46) with $c_1 = c_2 = 1$, thus, requires that all three nonzero eigenvalues of Eq. (46) be stable, which was sufficient but not necessary for stabilizability of Eq. (39). Computing the characteristic equation of Eq. (46) (with $c_1 = c_2 = 1$) we find

$$\lambda\Big\{\lambda^3 + d_2\lambda^2 + (\nu - \xi_1)(\xi_2 + \nu)\lambda + (\xi_1 - \nu)[e_2\alpha_2 x_1 - d_2(\xi_2 + \nu)]\Big\} = 0 \tag{47}$$

and we see that Eq. (47) differs from Eq. (42) only in the first- and zeroth-order coefficients of λ. This difference is significant because, as we near the desired value $x_1 \approx 1$, there exist achievable values of ν that render the first-order coefficient negative. For example, let $i_2 = 0.5$ and $i_3 = 0.3$. Then the first-order coefficient becomes $(\nu + 0.7)(\nu + 0.5)$, so that for all values of $\nu \in (-0.5, -0.7)$, the coefficient is negative, and the factorization is not stabilizable. This issue is not just of theoretical importance, for simulated attempts at SDRE nonlinear regulation of the gyrostat using Eq. (46) with $c_1 = c_2 = 1$ became numerically unstable when the factorization became unstabilizable. We mention this particular case because it corresponds to a natural SDC parameterization that arises from considering the gyrostat as a Hamiltonian system.[15] In this framework, the nonreduced state equations can be written $\dot{\boldsymbol{x}} = -\nabla\mathcal{H}^{\times}\boldsymbol{x}$, where ∇ is the gradient operator, $\mathcal{H}$ is an appropriate Hamiltonian, and the $(\cdot)^{\times}$ notation represents the skew symmetric matrix form of a vector.[14] Thus, selection of an appropriate SDC factorization can be nontrivial, and pointwise controllability issues should play a strong role in the process.

Finally, we need to consider detectability issues to ensure well posedness of the control. It has been shown in Ref. 12 that for H of the form

$$H = \mathrm{diag}(k_1, k_2, k_3, k_4) \tag{48}$$

with $k_i \in \mathcal{R}$ and $k_4 \neq 0$, detectability is guaranteed with A chosen according to Eq. (39), provided $x_1 > 0$. We investigate the effect of using two choices for H in the next section.

D. SDRE Nonlinear Regulator Simulation Results

In this section we give typical simulation results for the transverse spin-up maneuver using a sampled data implementation of the SDRE nonlinear regulator (which computes the SDRE solution at time t_k and applies the resulting constant control until the next sampling time t_{k+1}). For comparison purposes we also simulate the same maneuver using a small, constant torque. All simulations were performed in MATLAB/Simulink using Runge–Kutta, fourth-order integration. The sampling rate was 10 Hz, arbitrarily chosen to be implementable and to yield smooth-looking trajectories, and the integration step size was 0.01 s. In all simulations, we set $R = 1$. We give results for the five-state gyrostat model with damping. We choose a prolate spacecraft configuration, with inertia parameters $i_2 = 0.5$, $i_3 = 0.3$, and $i_1 = 0.2$. The initial condition is $(x_3, x_2, \mu_2, \nu) = [\sin(75\,\mathrm{deg}), 0, 0, 0.2\cos(75\,\mathrm{deg}) - 1]$, corresponding to the all-spun condition with a cone angle of 75 deg. We show state history results using

$$H = H_1 = \mathrm{diag}(0, 0, 0, 0.1) \quad \text{and} \quad H = H_2 = \mathrm{diag}(1, 1, 1, 0.1)$$

in Figs. 2 and 3, respectively. The penalty value of 0.1 on ν was selected to keep the control magnitude near 0.1, a reasonable value for spin-up maneuvers.[15]

We observe that for $H = H_1$, ν is driven to zero fairly quickly, but the x_2 and x_3 states have large initial oscillations, whereas for $H = H_2$, the initial oscillations of x_2 and x_3 are comparatively smaller and slower than for $H = H_1$. At the simulation end time, the x_i oscillations are comparatively larger for $H = H_2$, and ν takes much longer to go to zero for the $H = H_2$ case. These behaviors are intuitively satisfying in that they reflect the appropriate emphasis on controlling either ν only $(H = H_1)$ or a weighted combination of all of the states $(H = H_2)$. Although it is not apparent from Fig. 3 that ν actually does go to zero in the $H = H_2$ case, it is shown in Ref. 12 that it does. By leaving $k_4 = 0.1$ and decreasing the weights on the other states, it is possible to trade off performance in terms of obtaining quicker regulation of ν at the cost of larger initial deviations in x_2 and x_3. For comparison purposes we show

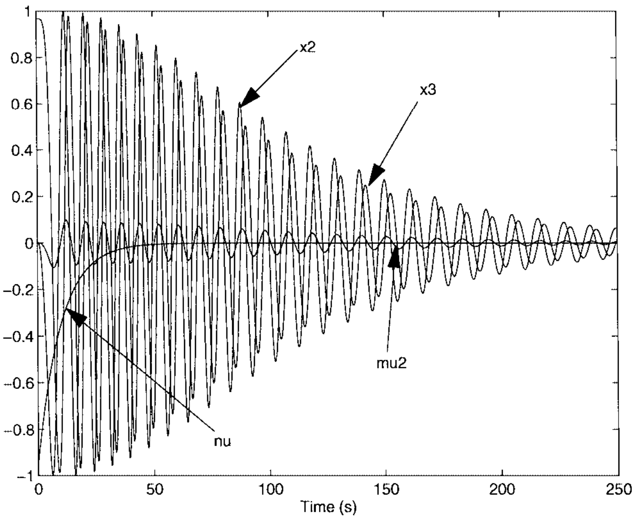

Fig. 2 Transverse SDRE state histories for five-state gyrostat, $H = H_1$.

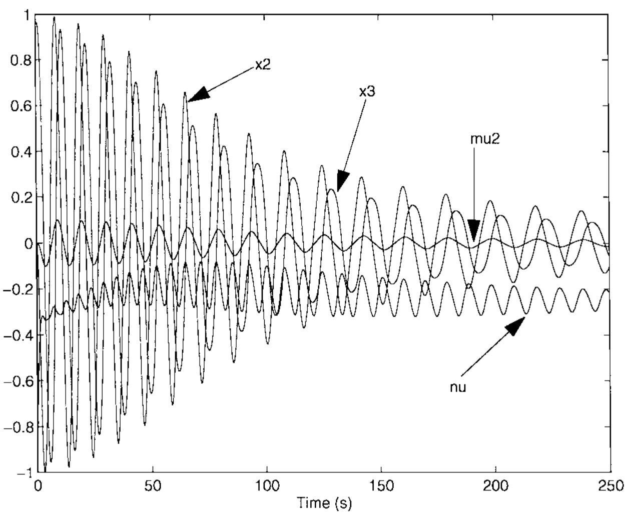

Fig. 3 Transverse SDRE state histories for five-state gyrostat, $H = H_2$.

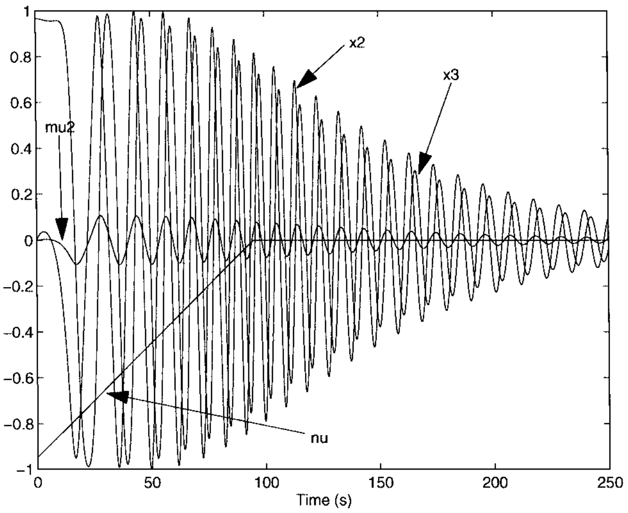

Fig. 4 Transverse state histories for five-state gyrostat, u = 0.01.

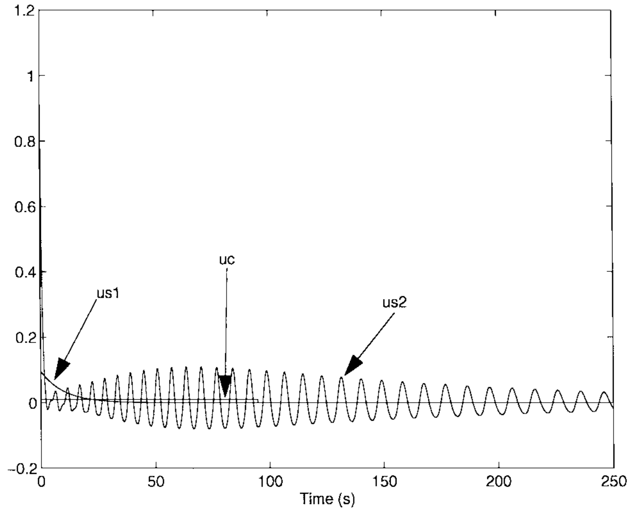

Fig. 5 Transverse control histories for five-state gyrostat.

state histories for the $u = 0.01$ constant torque case in Fig. 4. Note that the large initial oscillations in the x_2 and x_3 states continue to occur longer in the constant torque case, and their decay rates appear slower than in either of the SDRE cases. The initial decay of ν in the constant torque case is also slower than in both SDRE cases, but the final settling time is somewhere in between those of the SDRE cases. Finally, we show control histories for all three cases in Fig. 5. In the figure, uc is the constant torque control, whereas us1 and us2 are, respectively, the controls for the $H = H_1$ and $H = H_2$ cases. Significant features to be noticed from Fig. 5 are the smooth decay of us1 and the large initial value of us2, and its subsequent highly oscillatory nature. The high-energy content of us2 is again intuitively satisfying in that we expect large control effort to be needed to actively regulate the penalized x_2 and x_3 states because they are only indirectly affectable through μ.

VI. Summary and Conclusions

Nonlinear controllability of input-affine systems and controllability of state-dependent factorizations in terms of linear tests are not generally equivalent. Global pointwise controllability of a state-dependent factorization is sufficient to guarantee weak controllability on a neighborhood of the origin, and global controllability of both types holds when the input matrix function $B(\boldsymbol{x})$ has rank equal to the dimension of the state space n, for all $\boldsymbol{x}$. We have given sufficient conditions for controllability equivalency for second-order systems with constant B matrices, which may be extended to necessary conditions when B is not rank n. For higher-order systems, conditions guaranteeing equivalency become increasingly complex, due to differences between the Lie brackets that characterize nonlinear controllability and the $A^j B$ products in the factored controllability test.

Whereas factored controllability (plus observability) for all $\boldsymbol{x}$ is sufficient to guarantee global well posedness of SDRE-based control algorithms, it is, in general, not sufficient to guarantee true controllability outside some possibly small neighborhood of the origin, a fact which has significant ramifications for global stability. In fact, regardless of how well the SDRE algorithm operates, it can only affect the part of the state that is nonlinearly controllable, leading to a nonlinear stabilizability necessary condition for stability.

Successful transverse spin-up maneuvers of an axial gyrostat were performed using SDRE nonlinear regulation. Controllability analysis demonstrated nonlinear stabilizability in the region of interest. Also, a dynamics factorization, chosen on the basis of its strong pointwise controllability properties, guaranteed factored stabilizability where needed. The design problem showed that both types of controllability are separately important to successful application of the SDRE nonlinear regulation algorithm and demonstrated how the desired properties may be established.

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